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Exact multileg correlation functions for the dense phase of branching polymers in two dimensions

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We consider branching polymers on the planar square lattice with open boundary conditions and exactly calculate correlation functions of k polymer chains that connect two lattice sites with a large distance r apart for odd number of polymer chains k. We find that besides the standard power-law factor the leading term also has a logarithmic multiplier.

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Exact solutions of lattice statistical mechanical models, e.g., the two-dimensional Ising model, the Baxter model, etc., have played crucial role in our understanding of critical systems [1]. However, only a few lattice models can be solved exactly. Here we calculate exactly correlation functions for branching polymers on the planar square lattices when k polymer chains with odd k connect two lattice sites at a large distance r apart. We find that besides the standard power-law factor the leading term also has a logarithmic multiplier. Our work could inspire further studies on lattice models and logarithmic conformal field theories (CFT) [2].

The problem of branching polymers or spanning trees on the lattice is well known in statistical physics. In 1847, Kirchhoff [3] showed that the problem of spanning trees is related to the problem of resistance between the nodes of an electric circuit. Kirchhoff also proved a beautiful theorem that the number of spanning trees (branching or dense polymers) on the lattice of N sites is given by the principal minors of the $N\times N$ matrix of discrete Laplacian [4]. Later on it had been realized that the statistics of polymers is closely related to the statistics of spin models. Fortuin and Kasteleyn noticed that the partition function Z_N of the q-state Potts model can be represented as a dichromatic polynomial that continuously depends on q. Although the partition function of the model vanishes in the formal limit $q\to 0$ owing to zero mode of the discrete Laplacian (it is a polynomial in q with no zeroth power term), its derivative with respect to q does not and gives the partition function of dense lattice polymers (spanning trees) [5,6]; de Gennes [7] explained how the partition function of dilute polymers can be obtained from the partition function of the $O(n)$ model in the formal limit $n\to 0$. The nature of the phase transition from the high-temperature (dilute) phase to the low-temperature (dense) phase has been the subject of many investigations [8]. Nienhuis [9] showed how a particular $O(n)$ model on the hexagonal lattice can be mapped onto a Coulomb gas; the critical properties can be deduced from this mapping and the exponents obtained in this way are in good agreement with numerical estimates. Recently the model of branching polymers has found an alternate application for the sandpile model of self-organized criticality [10].

In spite of its long history the model of branching polymers has many open questions. One of them is about logarithmic corrections to the correlation functions in this model of the type shown in Fig. 1. Applying the Coulomb gas technique to the q-state Potts model, Saleur and Duplantier [11] found that asymptote of the correlation functions (which in their terminology can be considered as correlation function of two "k-leg operators" inserted into the plane and a distance r apart) has a power-law decay r^{-h_k} with critical exponents $h_k=gk^2/4-(4-g)^2/4g$, where g is related to q as: $q=2+2\cos(\pi g/2)$ for $g\in[2,4]$, and k is the number of polymers in a bunch (in Fig. 1, k is 3). In the limit $q\to 0$ (this corresponds to the phase when polymers cover the whole lattice densely) we have $g\to 2$ and $h_k=(k^2-1)/2$. Later, by studying the corresponding nonunitary CFT with the central charge $c=-2$, Gurarie [12] noticed that this theory admits the possibility of logarithmic corrections to the correlation functions, thus initiating the trend called logarithmic conformal field theories [2]. However, the structure and the properties of these theories are still rather poorly understood.

In this paper we consider branching polymers on the planar square lattice with open boundary conditions and calculate exactly k-leg correlation functions for odd number of legs k. We find that besides the critical exponent predicted by Saleur and Duplantier the leading term of the correlation function also has a logarithmic multiplier. We hope that this exact result could help to clarify the structure of nonunitary CFT with $c=-2$.

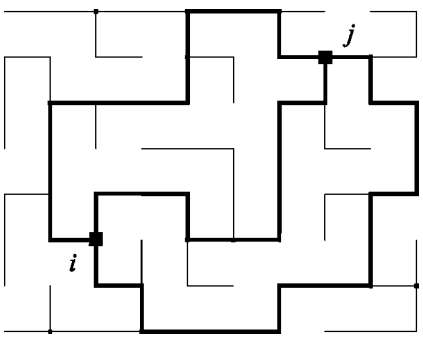

FIG. 1. Configuration of spanning tree on a planar square lattice that corresponds to the three-chain correlation between sites i and j.

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q-state Potts model. Here we briefly discuss $q \to 0$ limit in the q-state Potts model on lattice G with N sites and E bonds, whose Hamiltonian can be written as $H = J\Sigma_{\langle i,j\rangle}[1 - \delta(\sigma_i, \sigma_j)]$, where summation runs over all bonds $\langle i,j\rangle$ of the lattice G and σ_i, σ_j are spin variables with q components attached to the bond $\langle i,j\rangle$. Partition function of the model can be represented as [5,13]

$$Z(\{a\},q) = \sum_{\{\sigma\}} \exp\{-\beta H(\sigma)\} = \sum_{C\subseteq G} a^{|C|} b^{E-|C|} q^{\gamma(C)}. \quad (1)$$

Here $a = 1 - \exp(-\beta J)$, $b = 1 - a$, summation runs over all subgraphs C of G, $\gamma(C)$ is the number of connected components in C, and $|C|$ is the total number of bonds in C. To consider the limit $q \to 0$ in the partition function, we use the formal variables $a = \kappa S/(1+\kappa S)$ and $b = 1/(1+\kappa S)$ with $\kappa = \sqrt{q}$, then rewrite the partition function as

$$Z(\{S\},\kappa) = \frac{\kappa^{N+1}}{(1+\kappa S)^E} \sum_{C\subseteq G} S^{|C|} \kappa^{\omega(C)+\gamma(C)-1}, \quad (2)$$

where $\omega(C)$ is the total number of closed cycles in the subgraph C and Euler relation $\omega(C) + N = \gamma(C) + |C|$ has been used.

Now, let us rewrite the partition function as a series in power of κ,

$$Z(\{S\},\kappa) \sim \sum_{\substack{\omega(C)+\gamma(C)=1 \\ C\subseteq E}} S^{|C|} + \kappa \sum_{\substack{\omega(C)+\gamma(C)=2 \\ C\subseteq E}} S^{|C|} + \cdots. \quad (3)$$

The first one corresponds to subgraph C for which $\omega(C) + \gamma(C) = 1$. This is possible only when $\omega(C) = 0$ and $\gamma(C) = 1$. In other words, the first term in Eq. (3) corresponds to the sum over all spanning trees on the lattice. In the second term, summation is over all configurations for which $\omega(C) + \gamma(C) = 2$. Here we have two possibilities: either $\omega(C) = 0$, $\gamma(C) = 2$ or $\omega(C) = 1$, $\gamma(C) = 1$. The first one corresponds to two-component spanning trees while the second to configurations with only one cycle. It is interesting to note that both sets of configurations are dual to each other in the thermodynamic limit. In the limit $q \to 0$, only the first term survives. Thus we say that the model of spanning trees corresponds to the limit $q \to 0$ of the Potts model. The result, however, can be different if we are interested in correlation functions. In this case it is possible that the first term in the expansion (3) is identically equal to zero and the nontrivial contribution comes from some higher-order term of the series. As an example let us consider an average of $1 - \delta(\sigma_i, \sigma_j)$ in the Potts model, which is equal to 1, if spins σ_i and σ_j are different, and 0, if they coincide. In terms of a cluster model this combination of spin variables is equivalent to the indicator Δ_{ij} that the sites i and j belong to different connected components. That is why on the configurations that are represented by the only connected component (including spanning trees) this indicator vanishes identically. However, it would be nonzero on the set of two-component spanning trees. We conclude that although the partition function of the zero-component Potts model is equal to the number of spanning trees, to calculate correlation functions we need to consider more general configurations such as two-component spanning trees, etc. It is this fact which is responsible for the appearance of logarithmic corrections to correlation functions.

Before we calculate the correlation function of k polymer chains, we need to formulate the Kirchhoff theorem and remind the reader how to calculate local tree diagrams. To formulate the Kirchhoff theorem, we now recall some definitions from the graph theory. (a) A connected subgraph of a graph $\mathcal{L}$ which contains all its sites and has no cycles is a *spanning tree*. (b) A spanning tree with one site (the *root*) distinguished from all other sites by this very fact is called a *rooted* spanning tree. (c) Since the rooted spanning tree is a connected graph, there is a path from every site of it to the root. We may *orient* this path so that all its bonds will have arrows in the direction to the root. The tree property provides the consistency of this procedure for all bonds of the tree.

Kirchhoff theorem. We ascribe the weight x_{ij} to any bond of the graph $\mathcal{L}$, whose adjacent sites i and j are different from the root; the weight 1 to those bonds which are incident to the root. Then we define diagonal elements of the matrix of discrete Laplacian as a sum of weights of all bonds which are incident to the given site, $\Delta_{ii}(x) = \Sigma_n x_{in}$, where i is different from the root and the sum is over bonds which are incident to the site i. Off-diagonal elements of the matrix of discrete Laplacian are defined as $\Delta_{ij}(x) = -x_{ij}$ if sites i and j are adjacent and both are different from the root and zero otherwise. Then, the determinant of this matrix is a generating function of the rooted spanning trees on this graph. In particular, when $x_{ij} = 1$ for all i and j, this matrix coincides with the discrete Laplacian and its determinant gives the total number of rooted spanning trees [4].

The Kirchhoff theorem provides an effective tool to study tree diagrams. Here we breifly recall its principal ideas. Any modification of the weights of a finite number of lattice bonds is called a local defect of the lattice. For example, deleting the bonds or inserting additional ones can be considered as proper local defect. The difference between a discrete Laplacian of the new lattice Δ' and that of the initial one Δ is referred to as the defect matrix δ. Locality condition implies simply that only a finite number of the rows and columns of the defect matrix δ have nonzero elements.

Another important concept is a local tree diagram. We define it as a finite set of black and white arrows on the lattice bonds. Any spanning tree passing through all black arrows but not through the white ones is called compatible with this diagram. Given a local tree diagram, the problem we are interested in is to find the total number of compatible spanning trees. To solve this problem, we have to construct a matrix of the proper local defect by setting the weights of all bonds with white arrows to zero; whereas those with black arrows, to ϵ. Then, according to the Kirchhoff theorem the highest term of the polynomial $\Delta'(\epsilon)$ is just the number we are interested in. So the ratio $\mathcal{N}$ of the number of spanning trees compatible with a given diagram and the number of all trees on the lattice $\mathcal{L}$ is given by

$$\mathcal{N}(\delta) = \lim_{\epsilon\to\infty} \frac{\det \Delta'}{\epsilon^k \det \Delta} = \lim_{\epsilon\to\infty} \det(\mathbf{1} + G\delta)/\epsilon^k, \quad (4)$$

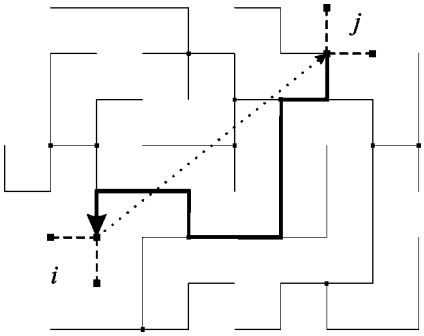

FIG. 2. One cycle configuration of spanning tree on a planar square lattice with an additional bond that connect sites i and j.

where k is the number of black arrows in the diagram, **1** is the unit matrix, and $G=\Delta^{-1}$ is the lattice Green's function which depends on the boundary-value problem for this Laplacian. For open boundary conditions at infinity the element of the matrix of Green's function between site i and site j has integral representation,

$$G(n,m) - G(0,0) = \frac{1}{8\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos n\beta \cos m\alpha - 1}{2 - \cos\beta - \cos\alpha} d\alpha d\beta. \tag{5}$$

Here the site i is chosen to have coordinates (0,0) on the planar square lattice, and site j coordinates (n,m). The asymptotic of the Green's function at large separations depends only on the distance between sites i and j ($r=\sqrt{n^2+m^2}\gg 1$). The main term of the expansion is logarithmic,

$$G(r) - G(0) = -\frac{1}{2\pi}\ln r - \frac{\gamma}{2\pi} - \frac{3\ln 2}{4\pi} + \cdots, \tag{6}$$

where γ is the *Euler* constant.

Priezzhev [14] observed that the Kirchhoff theorem can be generalized to count also some cyclic configurations by changing off-diagonal elements of the matrix δ while keeping its diagonal elements unchanged. For example, let us consider an arbitrary graph with matrix Δ and a set of k additional bonds $(i_1,j_1),\ldots,(i_k,j_k)$ not originally present in the graph with weights $-\epsilon_{i_1j_1},\ldots,-\epsilon_{i_kj_k}$, respectively. We may consider these as the only nonzero elements of the defect matrix δ. Then, the determinant of the new matrix $\Delta'=\Delta+\delta$ will generate not only spanning trees on the graph, but also all possible cycles on the graph running through these additional bonds. Each cycle being weighted with factor -1 and, hence, those configurations that have an even and odd number of cycles will have opposite signs. This observation gives us the possibility to use determinant formula to count also cyclic configurations.

Correlation matrix. Now we may calculate exactly the correlation functions considered by Saleur and Duplantier [11]. Namely, they were interested in the asymptotic behavior of such a correlation function when two sites of the lattice i and j at a distance r apart are connected by k polymer chains, when $r\to\infty$.

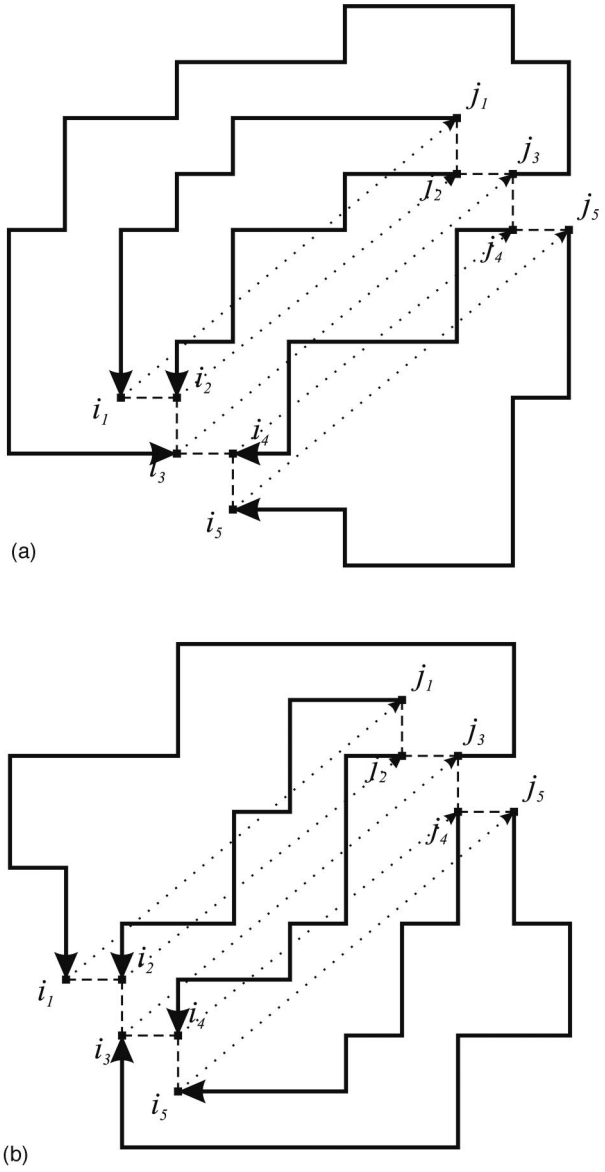

FIG. 3. Five cycles configuration (a) and one cycle configuration (b) of spanning tree on a planar square lattice with five additional bonds that connect sites $i_1,\ldots,i_5$ and $j_1,\ldots,j_5$.

Let us first consider the simplest case when $k=1$. In this case averaging over configurations that represent the first term of the expansion (3) leads to the following result: Since all sites of the lattice are connected by the spanning tree the average of this correlation function will be equal to 1 identically and does not depend on the distance r. This, of course, is the trivial example. Configurations that correspond to the second term produce a less trivial result. In this case a spanning tree may have two connected components and graphical representation of the corresponding configuration that produce nonzero contribution to the correlation function is shown in Fig. 2.

With the help of the generalized Kirchhoff theorem, we may easily count all such configurations. To this end, we add an additional bond to the lattice that connects the sites i and j and count only those configurations that have a cycle that

pass through the bond. It is clear that these are exactly the same configurations that contribute to the second term of the expansion (3). Defect matrix in this case can be written as $(\delta_1)_{ij}=-\epsilon$ and all other elements of the matrix are equal to zero. From Eq. (4), we obtain the ratio of the total number of such configurations and the number of all trees on the lattice (with a negative sign),

$$\mathcal{N}(\delta_1) = G_{ij} \sim \ln r, \tag{7}$$

i.e., averaging over two-component spanning trees results in logarithmic corrections to the correlation function as is given by asymptotic behavior of lattice Green's function (6).

Now let us consider a more general case of correlation functions with k polymer chains and add k additional bonds to the lattice and calculate the number of configurations with cycles that pass through all these additional bonds. One can verify that if the bonds are placed in a staircase manner as shown in Fig. 3, due to geometrical restrictions there are only two possibilities for the cycles to pass through the bonds. Either k different cycles pass every one through its own additional bond as shown in Fig. 3(a), or one cycle passes through all these k bonds as shown in Fig. 3(b). The former has a weight $(-1)^k$ while the latter (-1). Hence, these configurations will be counted with the same sign (-1) only when k is odd. This is the crucial observation for our further calculations.

In the limit of large separation of the sites i and j cyclic configurations will exactly correspond to those configurations when sites i and j are connected with a bunch made of k polymer chains. The elements of the defect matrix can be chosen as $(\delta_k)_{i_1 j_1}=\cdots=(\delta_k)_{i_k j_k}=-\epsilon$; all other matrix elements are 0. Now calculating the determinant (4) we obtain

$$\mathcal{N}(\delta_k) = \sum_P (-1)^{\sigma(P)} G_{i_1 j_{P(1)}} G_{i_2 j_{P(2)}} \cdots G_{i_k j_{P(k)}}, \tag{8}$$

where summation is done over all permutations P of the set $(j_1,\ldots,j_{2k-1})$ and $\sigma(P)$ is equal to 1 if permutation is odd and 0, otherwise.

To obtain more detailed information about the correlation functions, we have to study their asymptotic behavior at large separations $r \gg 1$. We may calculate such an asymptotic as follows. First, lattice Green's function depends only on the distances between sites i and j. If we denote coordinates of the difference on the planar square lattice as (n,m), from, e.g., Fig. 3, we can write down explicit expressions for the Green's functions (5) as follows: $G_{i_1 j_1}=G(n,m)$, $G_{i_1 j_2}=G(n,m-1)$, $G_{i_1 j_3}=G(n+1,m-1)$, $G_{i_1 j_4}=G(n+1,m-2),\ldots$. Then we consider Taylor expansions of these formulas at large separations when n, $m \gg 1$,

$$G_{i_1 j_2} = G(n,m-1) = G(n,m) - \partial_m G(n,m) + \ldots, \tag{9}$$

and take into account logarithmic asymptotic behavior of the lattice Green's function (6). We keep as many terms in this expansion as necessary to get first nonzero contributions in Eq. (8) to obtain the leading term of the asymptotic expansion of the correlation functions

$$\mathcal{N}(\delta_k) \sim r^{(1-k^2)/2} \ln r, \tag{10}$$

where $k=1,3,5,\ldots$.

This is the main result of the paper which is obtained analytically without any reference to the Coulomb gas technique or conformal field theory. It was essential for our calculation that the number of polymer chains that connect two sites of the lattice is odd. Unfortunately, the Kirchhoff theorem is not applicable for the analysis of an even number of chains since in this case different configurations are counted with opposite signs. At this moment we do not see any way around this problem. We believe this problem deserves further investigation. Also, it would be of interest to analyze higher-order terms of the expansion above as it may also contain higher powers of logarithms.

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